Notions of uniform manifolds

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Let M be an n-dimensional manifold; the assumed smoothness will be clear from the context. We are interested in uniformity properties of M when it is noncompact. These can be formulated in different ways, e.g. in terms of bounded geometry when a Riemannian metric g is present. If no such metric is (canonically) available, it may be more natural to express uniformity in terms of the atlas and its chart transition maps. We shall formulate various definitions of uniformity and investigate their relations.

We follow [Eic91a] to define bounded geometry.

Definition 1 (Bounded geometry)**.** *We say that a complete, finite-dimensional Riemannian manifold (M, g) has k-th order bounded geometry when the following conditions are satisfied:*

(I) *the global injectivity radius $r_{inj}(M) = \inf_{x \in M} r_{inj}(x)$ is positive, $r_{inj}(M) > 0$;*

($\mathbf{B}_k$) *the Riemannian curvature R and its covariant derivatives up to k-th order are uniformly bounded,*

$$\forall\, 0 \le i \le k \colon \sup_{x \in M} \|\nabla^i R(x)\| < \infty,$$

with operator norm of $\nabla^i R(x)$ as an element of the tensor bundle over $x \in M$.

This formulation is also called *coordinate-free* bounded geometry. It is shown in [Eic91b] (and in [Sch01] for manifolds with boundary) that this definition implies *coordinate-wise defined* bounded geometry, where condition $\mathbf{B}_k$ is replaced by $\mathbf{B}'_k$: there exists a radius $0 < r_0 < r_{\text{inj}}(M)$ such that on each normal coordinate chart of radius r_0, the metric coefficients g_{ij} and their derivatives up to order k are bounded by a global constant C_k. In [Roe88, Prop. 2.4] a proof is sketched that the converse also holds when $k = \infty$. Note that for finite k we would incur a loss of at least two degrees of differentiability, since the curvature is defined in terms of second derivatives of the metric.

Let us introduce a more general notion of uniformity, defined purely in terms of the atlas of a smooth manifold.

Definition 2 (Uniform manifold)**.** *We say that a complete manifold M with atlas $\mathcal{A} = \big\{(\phi_i \colon U_i \to \mathbb{R}^n) \mid i \in I\big\}$ is uniform of order $k \ge 1$ if*

(I) *there exists one uniform $\delta > 0$ such that for each $x \in M$ there exists a coordinate chart ϕ_i that covers a ball of radius δ around x, i.e.*

$$B(\phi_i(x); \delta) \subset \phi_i(U_i); \tag{1}$$

($\mathbf{B}_k$) *there is one global bound B_k such that all transition maps are uniformly bounded in C^k norm:*

$$\forall i, j \in I \colon \|\phi_j \circ \phi_i^{-1}\|_k \leq B_k. \tag{2}$$

Remark 3. *The C^0 part of the C^k bound restricts charts to have uniformly bounded image $\phi(U) \subset \mathbb{R}^n$. This is no loss of generality, since we can always break up a large chart and translate each part close to the origin. Alternatively, we could require the bound B_k for the derivatives of $(\phi_j \circ \phi_i^{-1})$ only.*

Definition 4 (Uniformly compatible atlases)**.** *Let $\mathcal{A}, \mathcal{A}'$ be two uniform C^k atlases for the manifold M. We say that these are uniformly compatible if the union $\mathcal{A} \cup \mathcal{A}'$ is again a uniform C^k atlas for M.*

Note that although this definition looks identical to the standard definition for (non-uniform) compatibility of atlases, it does implicitly depend on the parameters δ and B_k in Definition 2. The 'injectivity radius' δ of the combined atlas will at least be equal to the maximum of the radii of $\mathcal{A}$ and $\mathcal{A}'$, but the bound B_k of the combination may be larger than the maximum of their bounds. A maximal uniform atlas cannot be defined since it would require fixing a B_k, but this would mean that uniform compatibility of charts is not an equivalence relation anymore.

Classes of uniformly bounded C^k functions are defined as follows.

Definition 5 (Uniformly bounded C^k functions)**.** *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be uniform manifolds of order $k \geq 1$. Then we define the class $C_b^k(M; N)$ of uniformly bounded C^k functions to consist of those functions $f \in C^k(M; N)$ for which there exists a bound $C > 0$ such that the coordinate representations satisfy*

$$\|\psi \circ f \circ \phi^{-1}\|_k \leq C \tag{3}$$

for all charts $\phi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ (where defined).

Note that this class of functions is closed under composition, and under and multiplication e.g. when $N = \mathbb{R}$.

It turns out that definitions 1 and 2 are equivalent in the following sense.

Theorem 6. *Let $(M, \mathcal{A})$ be a uniform manifold. Then there exists a metric g such that (M, g) has bounded geometry and g induces an atlas $\mathcal{A}'$ of normal coordinate charts which is uniformly compatible again with $\mathcal{A}$.*

The order of smoothness may decrease by a small amount in the process $\mathcal{A} \rightsquigarrow g \rightsquigarrow \mathcal{A}'$. We shall work through a number of lemmas to prove this result.

Lemma 7. *Let (M, g) be a Riemannian manifold of k-th order (coordinate-free defined) bounded geometry with $k \geq 2$. Then M is a uniform manifold of order $k - 1$ with the preferred atlas given by the normal coordinate charts of some radius $\delta > 0$.*

See [Eld13, Lem. 2.6] for a proof. This result shows that the atlas of normal coordinate charts that arises from a manifold (M, g) of bounded geometry is itself uniform.

Remark 8. *We need not necessarily add the normal coordinate charts centered around all points $x \in M$ to the preferred atlas. If we instead constructed a (uniformly locally finite) cover as in [Eld13, Lem. 2.6] with normal coordinate balls of size δ_2, such that the balls of a fixed size $\delta_1 < \delta_2$ already cover M, then this satisfies Definition 2 with $\delta = \delta_2 - \delta_1$.*

We introduce some notation and intermediate results based on [Eld13, Sect. 2.1]. For any point $x \in M$, ϕ_x will denote a chosen coordinate chart that satisfies condition **I** of Definition 2. Then we define the open neighborhood

$$B_x(\delta') = \phi_x^{-1}\big(B(\phi_x(x); \delta')\big) \subset M \tag{4}$$

as the preimage of the ball of radius $\delta' \leq \delta$ around $\phi_x(x)$. Note that $B_x(\delta')$ need not be a ball in another coordinate chart ϕ, but since $\mathrm{D}(\phi \circ \phi_x^{-1})$ and its inverse are bounded by B_k, we have that

$$B\big(\phi(x); \delta'/B_k\big) \subset \phi\big(B_x(\delta)\big) \subset B\big(\phi(x); B_k\,\delta'\big) \tag{5}$$

insofar these lie within the image of ϕ, that is, coordinate chart transformations deform balls only boundedly so. In other words: local distances induced by the Euclidean distance in each of the charts are equivalent up to a factor B_k. Then we have the following result.

Lemma 9 (Uniformly locally finite cover)**.** *Let M be a uniform manifold of order $k \geq 1$ with parameters δ and B_k.*

Then for $0 < \delta_1 < \delta_2 < \delta$ small enough, M has a countable cover $\big\{B_{x_i}(\delta_2)\big\}_{i\geq 1}$ such that

1. *the sets $B_{x_i}(\delta_1)$ already cover M;*
2. *$\forall\, j \neq i\colon x_j \notin B_{x_i}(\delta_1/B_k)$;*
3. *there exists an explicit global bound $K \in \mathbb{N}$ such that for each $x \in M$ the neighborhood $B_x(\delta_2)$ intersects at most K of the $B_{x_i}(\delta_2)$.*

We follow the proof of [Eld13, Lem. 2.16] with appropriate modifications to replace the metric setting there.

Proof. Assume δ_1, δ_2 fixed, these will be determined later. Let $\{M_k\}_{k\geq 1}$ be an exhaustion of M by compact sets. Cover M_k by a finite sequence of balls $B_{x_i}(\delta_1)$ that extend the sequence covering M_{k-1}, as follows: choose a point $x \in M_k$ that is not covered yet, and add $B_x(\delta_1)$ to the sequence. This sequence is finite, for if it were infinite, it would have a converging subsequence $x_{i_j} \to \bar{x} \in M_k$. This is a contradiction since then $\|\phi_{\bar{x}}(x_{i_j}) - \phi_{\bar{x}}(\bar{x})\| \to 0$ and uniform equivalence of the Euclidean distances in charts now implies that the distance between points $x_{i_{j'}}$ in charts $\phi_{x_{i_j}}$ must converge to zero as $j, j' \to \infty$. This contradicts the assumption that new points x_i are not being covered yet. Since $x_j \notin B_{x_i}(\delta_1)$ for $i < j$ implies that $x_i \notin B_{x_j}(\delta_1/B_k)$, the limit of these sequences satisfies the first two claims of the lemma.

For the third claim let $x \in M$ be arbitrary. By (5) any ball $B_{x_i}(\delta_2)$ that intersects $B_x(\delta_2)$ must be completely contained in $B_x\big((1+2B_k)\delta_2\big)$, where we set $\delta_2 < \delta/(1+2B_k)$ to have this well-defined. In the chart ϕ_x, each $B_{x_i}(\delta_2)$ occupies an exclusive set $B(\phi_x(x_i); \delta_2/B_k)$ with respect to any of the other $B_{x_{i'}}(\delta_2)$ balls. Thus, by considering volume estimates in the chart ϕ_x we obtain

$$K \leq \frac{\big((1+2B_k)\delta_2\big)^n}{(\delta_2/B_k)^n} \leq \big(\sqrt{3}\,B_k\big)^{2n} \tag{6}$$

using that $B_k \geq 1$. $\square$

The following lemma is a straightforward adaptation of [Eld13, Lem. 2.17].

Lemma 10 (Uniform partition of unity)**.** *Let M be a uniform manifold with a uniformly locally finite cover with $0 < \delta_1 < \delta$ as per Lemma 9.*

Then there exists a partition of unity by functions $\chi_i \in C_b^k(B_{x_i}(\delta_2); [0,1])$ subordinate to this cover. There is a global bound on the C^k norm of all the functions χ_i.

Proof of Theorem 6. We first adapt a standard method to construct a Riemannian metric g, and then prove both that this metric has bounded geometry and that its normal coordinate chart atlas $\mathcal{A}'$ is uniformly compatible with the original uniform atlas $\mathcal{A}$.

Let $\{(\phi_i, B_{x_i}(\delta_2))\}_{i\in\mathbb{N}}$ be a uniformly locally finite cover of M with subordinate partition of unity by functions χ_i. Let g_e denote the Euclidean metric on $\mathbb{R}^n$. We define the metric

$$g = \sum_{i\in\mathbb{N}} \chi_i \, \phi_i^*(g_e). \tag{7}$$

Note that $g \in C_b^{k-1}$ since at most K terms in the sum are non-zero and each term is a composition and product of C_b^{k-1} functions $\chi_i, \phi_i, \mathrm{D}\phi_i$ and g_e; it is a sum of positive-definite bilinear forms, hence again positive-definite and invertible everywhere. Let $\phi \in \mathcal{A}$ be a coordinate chart and $v \in \mathbb{R}^n$ with $\|v\| = 1$, then we have a lower bound

$$\begin{aligned}
(\phi_* g)(v,v) &= \sum_{i\in\mathbb{N}} (\chi_i \circ \phi^{-1}) \cdot (\phi_i \circ \phi^{-1})^*(g_e)(v,v) \\
&= \sum_{i\in\mathbb{N}} (\chi_i \circ \phi^{-1}) \, \|\mathrm{D}(\phi_i \circ \phi^{-1})\, v\|^2 \\
&\geq \frac{1}{B_k^2},
\end{aligned}$$

hence g^{-1} is bounded by B_k^2 in any chart. From the expression of the derivatives of g^{-1} in terms of g^{-1} itself and derivatives of g, it follows that $g^{-1} \in C_b^{k-1}$; thus, also the Christoffel symbols satisfy $\Gamma \in C_b^{k-2}$ and this also proves that condition $\mathbf{B}_{k-3}$ of coordinate-free bounded geometry is satisfied.

To prove condition $\mathbf{I}$ of Definition 1 that g has a finite injectivity radius, and finally that the original atlas $\mathcal{A}$ and the atlas $\mathcal{A}'$ of normal coordinate charts are uniformly compatible, we consider coordinate transition maps from charts $\exp_x^{-1} \in \mathcal{A}'$ to $\phi \in \mathcal{A}$. Since exp is defined through the time-one geodesic flow, we can view a transition map

$$\phi^{-1} \circ \exp_x \colon \mathrm{T}_x M \to \mathbb{R}^n \tag{8}$$

as a local coordinate expression of $\exp_x = \pi \circ \Upsilon^1|_{\mathrm{T}_x M}$, where Υ^t denotes the geodesic flow on $\mathrm{T}M$. This flow is defined by the differential equation

$$\begin{aligned}
\dot{x}^i &= v^i, \\
\dot{v}^i &= \Gamma(x)^i_{jk} v^j v^k,
\end{aligned} \tag{9}$$

with respect to coordinates (x^i, v^j) on $\mathrm{T}M$ induced by ϕ. The flow Υ^t preserves $\|v\|^2 = g_x(v,v)$ and g, g^{-1} are uniformly bounded, so also the coordinate expressions v^i are uniformly bounded by $B_k^2\,\delta'$ when the initial value $v(0)$ is bounded by δ'. Since Γ is bounded also, we have that (9) approximates the system

$$\begin{aligned} \dot{x}^i &= v^i, \\ \dot{v}^i &= 0, \end{aligned} \tag{10}$$

when $\|v(0)\| \leq \delta'$ is small. Note that (10) induces the 'Euclidean exponential map' $\widetilde{\exp}_x(v) = x + v$ with respect to the chart ϕ. Thus by uniform dependence of a flow on the vector field (see [Eld13, Thm. A.6]) it follows that $\phi^{-1} \circ \exp_x = \phi^{-1} \circ \pi \circ \Upsilon^1(x, \cdot)$ approximates the identity map on $\mathbb{R}^n$ in C^{k-2} norm for sufficiently small δ'. Hence the transition maps are uniform C^{k-2} diffeomorphisms and thus the altases $\mathcal{A}$ and $\mathcal{A}'$ are uniformly compatible. This also shows that the injectivity radius of g satisfies $r_{\mathrm{inj}}(M) \geq \delta'$. $\square$

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